

THE RADIUS OF UNIFORM CONVEXITY OF BESSEL FUNCTIONS

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ABSTRACT. In this paper, we determine the radius of uniform convexity for three kinds of normalized Bessel functions of the first kind. In the mentioned cases the normalized Bessel functions are uniformly convex on the determined disks. Moreover, necessary and sufficient conditions are given for the parameters of the three normalized functions such that they to be uniformly convex in the open unit disk. The basic tool of this study is the development of Bessel functions in function series.

1. INTRODUCTION

It is well known that the concepts of convexity, starlikeness, close-to-convexity and uniform convexity including necessary and sufficient conditions, have a long history as a part of geometric function theory. In 1945, Pólya and Szegő [33] found the necessary and sufficient conditions of convexity and starlikeness for analytic functions which were further generalized by Royster [37] and Bernardi [9]. After that several authors contributed to this literature by doing generalizations of the previously developed criteria and also by introducing some new ones, for details, see [12, 14, 30]. Significant contributions to the same were made by Mocanu [20, 21], Obradović [25] and Owa et.al [29] by introducing more applicable criteria for close-to-convexity, convexity and starlikeness. Tuneski [45] used the method of differential subordination to find the conditions for starlikeness of analytic functions. In 1993, Rønning [35] determined necessary and sufficient conditions of analytic functions to be uniformly convex in the open unit disk, while in 2002 Ravichandran [34] also presented more simple criteria for uniform convexity. Silverman [40] investigated the properties of functions defined in terms of the quotient of analytic representations of convex and starlike functions which was then improved by Obradović and Tuneski [26] and Tuneski [46]. Recent works on certain criteria of convexity and starlikeness can be found in [22, 23, 24, 47].

On the other hand, one of the most important applications of the concepts of convexity, starlikeness, close-to-convexity and uniform convexity is to find the necessary and sufficient condition of theirs for hypergeometric and Bessel functions. In 1961, Merkes and Scott [18] investigated the starlikeness and univalence of Gaussian hypergeometric functions by using continued-fraction representations, while in 1986 Ruscheweyh and Singh [36] obtained the exact order of starlikeness by using the same technique. Owa and Srivastava [28] investigated the geometric properties of generalized hypergeometric functions using the well known Jack's lemma. Miller and Mocanu [19] employed the method of differential subordinations to investigate the local univalence, starlikeness and convexity of certain hypergeometric functions. Silverman [39] in 1993 also investigated the starlikeness and convexity of Gaussian hypergeometric functions, while in 1998 and 2001 Ponnusamy and Vuorinen [31, 32] presented some generalizations of the results of Miller and Mocanu, and determined conditions of close-to-convexity of confluent (or Kummer) and Gaussian hypergeometric functions, respectively. Küstner [17] by using among others the continued

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fraction of C.F. Gauss determined the order of convexity and starlikeness of hypergeometric functions. Many authors have determined the necessary and sufficient conditions for the hypergeometric functions to be uniform convexity [15, 41, 42]. An extensive bibliography and history of convexity, starlikeness and close-to-convexity for Bessel functions can be found in the two sections of Chap. 3.

Let $U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ denote the disk of radius r and center z_0 . We denote by $U(r) = U(0, r)$ and by $U = U(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$. Let $(a_n)_{n \geq 2}$ be a sequence of complex numbers with

$$d = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \geq 0, \text{ and } r_f = \frac{1}{d}.$$

If $d = 0$ then $r_f = +\infty$. The power series

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

defines an analytic function $f : U(r_f) \rightarrow \mathbb{C}$.

For $r \in (0, r_f)$ we say that the function f is starlike in the disk $U(r) = \{z \in \mathbb{C} : |z| < r\}$ if f is univalent in $U(r)$, and $f(U(r))$ is a starlike domain with respect to 0 in \mathbb{C} . Regarding the starlikeness of the function f the following equivalency holds

$$f \text{ is starlike in } U(r) \text{ if and only if } \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, \quad z \in U(r).$$

We define by

$$r_f^* = \sup \left\{ r \in (0, r_f) : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, \quad z \in U(r) \right\}$$

the radius of starlikeness of the function f .

The radius of convexity is defined in a similar manner. We say that a function f of the form (1.1) is convex if f is univalent and $f(U(r))$ is a convex domain in \mathbb{C} . An analytic description of this definition is

$$f \in \mathcal{A} \text{ is convex if and only if } \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in U(r).$$

The radius of convexity of the function f is defined by

$$r_f^c = \sup \left\{ r \in (0, r_f) : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in U(r) \right\}.$$

We will give a few definitions and results in the next section which we will use further on to determine the radius of uniform convexity.

2. PRELIMINARIES

In the following we deal with the class of the uniformly convex functions. Goodman in [13] introduced the concept of uniform convexity for functions of the form (1.1). A function f is said to be uniformly convex in $U(r)$ if f is of the form (1.1), it is convex, and has the property that for every circular arc γ contained in $U(r)$, with center ς , also in $U(r)$, the arc $f(\gamma)$ is convex. An analytic description of the uniformly convex functions is given in the next theorem, which is a slight modification of Theorem 1 from [35].

Theorem 2.1. *Let f be a function of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and analytic in the disk $U(r)$. The function f is uniformly convex in the disk $U(r)$ if and only if*

$$(2.1) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in U(r).$$

This theorem makes possible to determine the radius of uniform convexity of Bessel functions. The radius of uniform convexity is defined by

$$r_f^{uc} = \sup \left\{ r \in (0, r_f) : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in U(r) \right\}.$$

In order to prove the main results later on, we need the following lemma.

Lemma 2.1. *i. If $a > b > r \geq |z|$, and $\lambda \in [0, 1]$, then*

$$(2.2) \quad \left| \frac{z}{b-z} - \lambda \frac{z}{a-z} \right| \leq \frac{r}{b-r} - \lambda \frac{r}{a-r}.$$

Very simple consequences of this inequality are the followings

$$(2.3) \quad \operatorname{Re} \left(\frac{z}{b-z} - \lambda \frac{z}{a-z} \right) \leq \frac{r}{b-r} - \lambda \frac{r}{a-r}$$

and

$$(2.4) \quad \operatorname{Re} \left(\frac{z}{b-z} \right) \leq \left| \frac{z}{b-z} \right| \leq \frac{r}{b-r}.$$

ii. If $b > a > r \geq |z|$, then

$$(2.5) \quad \left| \frac{1}{(a+z)(b-z)} \right| \leq \frac{1}{(a-r)(b+r)}.$$

Proof. *i.* According to the maximum principle for harmonic functions we have to prove inequality (2.2) only in case $z = re^{i\theta}$. In this case the inequality is equivalent to

$$(2.6) \quad \left| \frac{1}{b - re^{i\theta}} - \lambda \frac{1}{a - re^{i\theta}} \right| \leq \frac{1}{b-r} - \lambda \frac{1}{a-r}.$$

Denoting $a_1 = \frac{a}{r}, b_1 = \frac{b}{r}$ inequality (2.6) becomes

$$(2.7) \quad \left| \frac{1}{b_1 - e^{i\theta}} - \lambda \frac{1}{a_1 - e^{i\theta}} \right| \leq \frac{1}{b_1 - 1} - \lambda \frac{1}{a_1 - 1}, \quad a_1 > b_1 > 1, \quad \theta \in [0, 2\pi].$$

Let the function $\varphi : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$\varphi(\lambda) = \left(\frac{1}{b_1 - 1} - \lambda \frac{1}{a_1 - 1} \right)^2 - \left| \frac{1}{b_1 - e^{i\theta}} - \lambda \frac{1}{a_1 - e^{i\theta}} \right|^2.$$

By denoting $t = \cos \theta$, the function φ can be rewritten in the following form

$$\begin{aligned} \varphi(\lambda) &= \left(\frac{1}{b_1 - 1} - \lambda \frac{1}{a_1 - 1} \right)^2 - \frac{1}{b_1^2 - 2b_1t + 1} - \lambda^2 \frac{1}{a_1^2 - 2a_1t + 1} \\ &\quad + 2\lambda \frac{a_1b_1 - (a_1 + b_1)t + 1}{(a_1^2 - 2a_1t + 1)(b_1^2 - 2b_1t + 1)}. \end{aligned}$$

We have that

$$\begin{aligned} \varphi'(\lambda) &= 2 \left(\frac{1}{b_1 - 1} - \lambda \frac{1}{a_1 - 1} \right) \left(-\frac{1}{a_1 - 1} \right) - 2\lambda \frac{1}{a_1^2 - 2a_1t + 1} \\ &\quad + 2 \frac{a_1b_1 - (a_1 + b_1)t + 1}{(a_1^2 - 2a_1t + 1)(b_1^2 - 2b_1t + 1)} \end{aligned}$$

and

$$\varphi''(\lambda) = 2 \frac{1}{(a_1 - 1)^2} - 2 \frac{1}{a_1^2 - 2a_1t + 1} > 0.$$

Thus φ' is strictly increasing on $[0, 1]$, and consequently if $\varphi'(1) < 0$, then $\varphi'(\lambda) < 0$, $\lambda \in [0, 1]$.

Some calculations lead to

$$\varphi'(1) = 2(a_1 - b_1) \left(\frac{b_1 - t}{(a_1^2 - 2a_1t + 1)(b_1^2 - 2b_1t + 1)} - \frac{1}{(a_1 - 1)^2(b_1 - 1)} \right).$$

If

$$m(t) = \frac{b_1 - t}{(a_1^2 - 2a_1t + 1)(b_1^2 - 2b_1t + 1)},$$

then

$$\varphi'(1) = 2(a_1 - b_1) (m(t) - m(1)).$$

Since

$$m'(t) = \frac{(a_1^2 - 2a_1t + 1)(b_1^2 - 1) + 2a_1(b_1^2 - 2b_1t + 1)(b_1 - t)}{(a_1^2 - 2a_1t + 1)^2(b_1^2 - 2b_1t + 1)^2} > 0,$$

it follows that $m(t) \leq m(1)$ and consequently $\varphi'(1) < 0$, and $\varphi'(\lambda) < 0$, $\lambda \in [0, 1]$. This implies that φ is strictly decreasing and $\varphi(\lambda) \geq \varphi(1)$ or equivalently

$$\begin{aligned} & \left(\frac{1}{b_1 - 1} - \lambda \frac{1}{a_1 - 1} \right)^2 - \left(\frac{1}{b_1 - e^{i\theta}} - \lambda \frac{1}{a_1 - e^{i\theta}} \right)^2 \\ & \geq \left(\frac{1}{b_1 - 1} - \frac{1}{a_1 - 1} \right)^2 - \left(\frac{1}{b_1 - e^{i\theta}} - \frac{1}{a_1 - e^{i\theta}} \right)^2, \quad \theta \in [0, 2\pi]. \end{aligned}$$

Thus in order to prove (2.7) we have to show that

$$(2.8) \quad \frac{1}{b_1 - 1} - \frac{1}{a_1 - 1} \geq \left| \frac{1}{b_1 - e^{i\theta}} - \frac{1}{a_1 - e^{i\theta}} \right|, \quad a_1 > b_1 > 1, \quad \theta \in [0, 2\pi]$$

or equivalently

$$(2.9) \quad \frac{1}{(a_1 - 1)(b_1 - 1)} \geq \frac{1}{|(a_1 - e^{i\theta})(b_1 - e^{i\theta})|}, \quad a_1 > b_1 > 1, \quad \theta \in [0, 2\pi].$$

Since the inequalities $|a_1 - e^{i\theta}| \geq a_1 - 1$ and $|b_1 - e^{i\theta}| \geq b_1 - 1$ for $a_1 > b_1 > 1$, $\theta \in [0, 2\pi]$ holds we get (2.9) and thus (2.8). We mention that (2.3) and (2.4) have been proved in [3] using a direct method.

ii. According to the maximum principle it is enough to prove the inequality (2.5) in case of $z = re^{i\theta}$, that is

$$(2.10) \quad \left| \frac{1}{(a + re^{i\theta})(b - re^{i\theta})} \right| \leq \frac{1}{(a - r)(b + r)}.$$

Denoting $\alpha = \frac{a}{r}$ and $\beta = \frac{b}{r}$, the inequality (2.10) can be rewritten as follows

$$(2.11) \quad \left| \frac{1}{(\alpha + e^{i\theta})(\beta - e^{i\theta})} \right| \leq \frac{1}{(\alpha - 1)(\beta + 1)},$$

where $\beta > \alpha > 1$. We will prove the inequality (2.11). If $t = \cos \theta$, then this inequality will be equivalent to

$$(2.12) \quad (\beta^2 - 2\beta t + 1)(\alpha^2 + 2\alpha t + 1) \geq (\beta + 1)^2(\alpha - 1)^2, \quad \beta > \alpha > 1, \quad t \in [-1, 1].$$

In order to prove inequality (2.12) we define the function

$$u : [-1, 1] \rightarrow \mathbb{R}, \quad u(t) = (\beta^2 - 2\beta t + 1)(\alpha^2 + 2\alpha t + 1).$$

Since $u''(t) = -8\alpha\beta < 0$, $t \in [-1, 1]$ it follows that u is a concave mapping, and consequently

$$u(t) \geq \min\{u(1), u(-1)\} = u(-1) = (\beta + 1)^2(\alpha - 1)^2.$$

Thus the proof of the inequality (2.5) is done. □

3. MAIN RESULTS

The Bessel function of the first kind of order ν is defined by [27, p. 217]

$$J_\nu(z) = \sum_{n \geq 0} \frac{(-1)^n}{n! \Gamma(n + \nu + 1)} \left(\frac{z}{2}\right)^{2n + \nu}.$$

In this paper we deal with the following normalized forms

$$f_\nu(z) = (2^\nu \Gamma(\nu + 1) J_\nu(z))^{\frac{1}{\nu}} = z - \frac{1}{4\nu(\nu + 1)} z^3 + \dots, \quad \nu \neq 0,$$

$$g_\nu(z) = 2^\nu \Gamma(\nu + 1) z^{1-\nu} J_\nu(z) = z - \frac{1}{4(\nu + 1)} z^3 + \frac{1}{32(\nu + 1)(\nu + 2)} z^5 - \dots,$$

$$h_\nu(z) = 2^\nu \Gamma(\nu + 1) z^{1-\frac{\nu}{2}} J_\nu(\sqrt{z}) = z - \frac{1}{4(\nu + 1)} z^2 + \dots,$$

where $\nu > -1$. Observe that $f_\nu, g_\nu, h_\nu \in \mathcal{A}$. We note that

$$f_\nu(z) = \exp \left(\frac{1}{\nu} \log (2^\nu \Gamma(\nu + 1) J_\nu(z)) \right),$$

where \log represents the principal branch of the logarithm.

Using the proved inequalities we will determine the smallest value δ such that the inequality $\nu \geq \delta$ implies the uniform convexity in U of different normalized Bessel functions. The radius of uniform convexity of the Bessel functions of the first kind are also determined. These two questions are closely connected.

Here and in the sequel I_ν denotes the modified Bessel function of the first kind and order ν . Note that $I_\nu(z) = i^{-\nu} J_\nu(iz)$ and $I_\nu(\sqrt{z}) = (-1)^{-\frac{\nu}{2}} J_\nu(\sqrt{-z})$.

3.1. The radius of uniform convexity of normalized Bessel functions. As far as we know the first results regarding the starlikeness of Bessel functions have been given in [10] and [16]. These two papers initiated a research to study the univalence of Bessel functions and determine the radius of starlikeness for different kind of normalizations. These results raise questions about other geometric properties of Bessel functions, like convexity, uniform convexity and etc. Recently, Baricz et al. [2], Baricz and Szász [3] and Baricz et al. [4] obtained, respectively, the radius of starlikeness of order β , the radius of convexity of order β and the radius of α -convexity of order β for the functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$ in the case when $\nu > -1$. On the other hand, we know that if $\nu \in (-2, -1)$, then the Bessel function has exactly two purely imaginary conjugate complex zeros, and all the other zeros are real (see [48] p. 483). Thus in order to solve the above radius problems in case $\nu \in (-2, -1)$, the method which has been used in [2, 3, 4] is not applicable directly. In [44], Szász investigated the radius of starlikeness of order β for the functions $g_\nu(z)$ and $h_\nu(z)$ in the case when $\nu \in (-2, -1)$ by using some inequalities. Baricz and Szász [5] obtained the radius of convexity of order β for the functions $g_\nu(z)$ and $h_\nu(z)$ in the case when $\nu \in (-2, -1)$. Very Recently, Deniz et al. [11] investigated the radius of α -convexity of order β for the functions $g_\nu(z)$ and $h_\nu(z)$ in the case when $(-2, -1)$. In the paper [1] the radius of starlikeness and the radius of convexity of q -Bessel functions have also been determined. In this section, we deal with the radius of uniform convexity for the normalized Bessel functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$ in the case when $\nu > -2$ ($\nu \neq -1$).

Theorem 3.1. *If $\nu > 0$, then the radius of uniform convexity of the function f_ν is the smallest positive root of the equation*

$$1 + 2 \frac{(r^2 - \nu^2)J_\nu(r)}{rJ'_\nu(r)} + 2 \left(1 - \frac{1}{\nu}\right) \frac{rJ'_\nu(r)}{J_\nu(r)} = 0.$$

Moreover $r^{uc}(f_\nu) < r^c(f_\nu) < j'_{\nu,1} < j_{\nu,1}$, where $j_{\nu,1}$ and $j'_{\nu,1}$ denote the first positive zeros of J_ν and J'_ν , respectively and $r^c(f_\nu)$ is the radius of convexity of the function f_ν .

Proof. Let $j_{\nu,n}$ and $j'_{\nu,n}$ are the n -th positive roots of J_ν and J'_ν , respectively. In [3, p.11] the following equality was proved

$$1 + \frac{zf''_\nu(z)}{f'_\nu(z)} = 1 - \left(\frac{1}{\nu} - 1\right) \sum_{n \geq 1} \frac{2z^2}{j_{\nu,n}^2 - z^2} - \sum_{n \geq 1} \frac{2z^2}{j_{\nu,n}'^2 - z^2}.$$

We will prove the theorem in two steps.

First suppose $\nu \geq 1$. In this case we will use the property of the zeros $j_{\nu,n}$ and $j'_{\nu,n}$, that interlace according to the inequalities

$$\nu \leq j'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < j_{\nu,2} < j'_{\nu,3} < \dots$$

Putting $\lambda = 1 - \frac{1}{\nu}$, inequality (2.3) implies $\operatorname{Re} \left(\frac{2z^2}{j_{\nu,n}^2 - z^2} - \left(1 - \frac{1}{\nu}\right) \frac{2z^2}{j_{\nu,n}'^2 - z^2} \right) \leq \frac{2r^2}{j_{\nu,n}'^2 - r^2} - \left(1 - \frac{1}{\nu}\right) \frac{2r^2}{j_{\nu,n}^2 - r^2}$, for $|z| \leq r < j'_{\nu,1} < j_{\nu,1}$, and we get

$$\begin{aligned} (3.1) \quad \operatorname{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)} \right) &= 1 - \sum_{n \geq 1} \operatorname{Re} \left(\frac{2z^2}{j_{\nu,n}^2 - z^2} - \left(1 - \frac{1}{\nu}\right) \frac{2z^2}{j_{\nu,n}'^2 - z^2} \right) \\ &\geq 1 - \sum_{n \geq 1} \left(\frac{2r^2}{j_{\nu,n}'^2 - r^2} - \left(1 - \frac{1}{\nu}\right) \frac{2r^2}{j_{\nu,n}^2 - r^2} \right) \\ &= 1 + \frac{rf''_\nu(r)}{f'_\nu(r)}. \end{aligned}$$

On the other hand if in the inequality (2.2) we replace z by z^2 and we put again $\lambda = 1 - \frac{1}{\nu}$, then it follows that $\left| \frac{2z^2}{j_{\nu,n}^2 - z^2} - \left(1 - \frac{1}{\nu}\right) \frac{2z^2}{j_{\nu,n}'^2 - z^2} \right| \leq \frac{2r^2}{j_{\nu,n}'^2 - r^2} - \left(1 - \frac{1}{\nu}\right) \frac{2r^2}{j_{\nu,n}^2 - r^2}$ provided that $|z| \leq r < j'_{\nu,1} < j_{\nu,1}$. Thus, we have

$$\begin{aligned} (3.2) \quad \left| \frac{zf''_\nu(z)}{f'_\nu(z)} \right| &= \left| \sum_{n \geq 1} \left(\frac{2z^2}{j_{\nu,n}^2 - z^2} - \left(1 - \frac{1}{\nu}\right) \frac{2z^2}{j_{\nu,n}'^2 - z^2} \right) \right| \\ &\leq \sum_{n \geq 1} \left| \frac{2z^2}{j_{\nu,n}^2 - z^2} - \left(1 - \frac{1}{\nu}\right) \frac{2z^2}{j_{\nu,n}'^2 - z^2} \right| \\ &\leq \sum_{n \geq 1} \left(\frac{2r^2}{j_{\nu,n}'^2 - r^2} - \left(1 - \frac{1}{\nu}\right) \frac{2r^2}{j_{\nu,n}^2 - r^2} \right) = -\frac{rf''_\nu(r)}{f'_\nu(r)}. \end{aligned}$$

In the second step we will prove that inequalities (3.1) and (3.2) hold in the case $\nu \in (0, 1)$ too. Indeed in the case $\nu \in (0, 1)$ the roots $0 < j'_{\nu,n} < j_{\nu,n}$ are real for every natural number n . Inequality (2.4) implies

$$\operatorname{Re} \frac{2z^2}{j_{\nu,n}'^2 - z^2} \leq \left| \frac{2z^2}{j_{\nu,n}'^2 - z^2} \right| \leq \frac{2r^2}{j_{\nu,n}'^2 - r^2}, \quad |z| \leq r < j'_{\nu,1} < j_{\nu,1}$$

and

$$\operatorname{Re} \frac{2z^2}{j_{\nu,n}^2 - z^2} \leq \left| \frac{2z^2}{j_{\nu,n}^2 - z^2} \right| \leq \frac{2r^2}{j_{\nu,n}^2 - r^2}, \quad |z| \leq r < j'_{\nu,1} < j_{\nu,1}.$$

Since $\frac{1}{\nu} - 1 > 0$, the previous inequalities imply that

$$\begin{aligned}
 (3.3) \quad \operatorname{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)} \right) &= 1 - \sum_{n \geq 1} \operatorname{Re} \left(\frac{2z^2}{j_{\nu,n}^2 - z^2} \right) - \left(\frac{1}{\nu} - 1 \right) \sum_{n \geq 1} \operatorname{Re} \left(\frac{2z^2}{j_{\nu,n}^2 - z^2} \right) \\
 &\geq 1 - \sum_{n \geq 1} \frac{2r^2}{j_{\nu,n}^2 - r^2} - \left(\frac{1}{\nu} - 1 \right) \sum_{n \geq 1} \frac{2r^2}{j_{\nu,n}^2 - r^2} \\
 &= 1 + \frac{rf''_\nu(r)}{f'_\nu(r)}.
 \end{aligned}$$

Now if in the second part of inequality (2.4) we replace z by z^2 , and b by $j'_{\nu,n}$ and by $j_{\nu,n}$, respectively, then it follows that $\left| \frac{2z^2}{j_{\nu,n}^2 - z^2} \right| \leq \frac{2r^2}{j_{\nu,n}^2 - r^2}$, and $\left| \frac{2z^2}{j_{\nu,n}^2 - z^2} \right| \leq \frac{2r^2}{j_{\nu,n}^2 - r^2}$, provided that $|z| \leq r < j'_{\nu,1} < j_{\nu,1}$.

These two inequalities and the condition $\frac{1}{\nu} - 1 > 0$, imply

$$\begin{aligned}
 (3.4) \quad \left| \frac{zf''_\nu(z)}{f'_\nu(z)} \right| &= \left| \sum_{n \geq 1} \left(\frac{2z^2}{j_{\nu,n}^2 - z^2} + \left(\frac{1}{\nu} - 1 \right) \frac{2z^2}{j_{\nu,n}^2 - z^2} \right) \right| \\
 &\leq \sum_{n \geq 1} \left| \frac{2z^2}{j_{\nu,n}^2 - z^2} \right| + \left(\frac{1}{\nu} - 1 \right) \sum_{n \geq 1} \left| \frac{2z^2}{j_{\nu,n}^2 - z^2} \right| \\
 &\leq \sum_{n \geq 1} \left(\frac{2r^2}{j_{\nu,n}^2 - r^2} + \left(\frac{1}{\nu} - 1 \right) \frac{2r^2}{j_{\nu,n}^2 - r^2} \right) = -\frac{rf''_\nu(r)}{f'_\nu(r)}.
 \end{aligned}$$

Finally from (3.1) and (3.2) we infer

$$(3.5) \quad \operatorname{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)} \right) - \left| \frac{zf''_\nu(z)}{f'_\nu(z)} \right| \geq 1 + 2\frac{rf''_\nu(r)}{f'_\nu(r)}, \quad |z| \leq r < j'_{\nu,1},$$

and (3.3), (3.4) also lead to the previous inequality. The equality holds if and only if $z = r$. Thus it follows

$$\inf_{|z| < r} \left[\operatorname{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)} \right) - \left| \frac{zf''_\nu(z)}{f'_\nu(z)} \right| \right] = 1 + 2\frac{rf''_\nu(r)}{f'_\nu(r)}, \quad r \in (0, j'_{\nu,1}).$$

The mapping $\psi_\nu : (0, j'_{\nu,1}) \rightarrow \mathbb{R}$ defined by

$$\psi_\nu(r) = 1 + 2\frac{rf''_\nu(r)}{f'_\nu(r)} = 1 - 2 \sum_{n \geq 1} \left(\frac{2r^2}{j_{\nu,n}^2 - r^2} - \left(1 - \frac{1}{\nu} \right) \frac{2r^2}{j_{\nu,n}^2 - r^2} \right)$$

is strictly decreasing, $\lim_{r \searrow 0} \psi_\nu(r) = 1$ and $\lim_{r \nearrow j'_{\nu,n}} \psi_\nu(r) = -\infty$. Thus it follows that the equation $1 + 2\frac{rf''_\nu(r)}{f'_\nu(r)} = 0$ has a unique root $r_0 \in (0, j'_{\nu,n})$ and $r_0 = r^{uc}(f_\nu)$. Since $1 + 2\frac{rf''_\nu(r)}{f'_\nu(r)} = 1 + 2\frac{rJ''_\nu(r)}{J'_\nu(r)} + 2\left(\frac{1}{\nu} - 1\right) \frac{rJ'_\nu(r)}{J_\nu(r)}$ and using the Bessel differential equation $z^2 J''_\nu(z) + zJ'_\nu(z) + (1 - \nu^2)J_\nu(z) = 0$, the proof is done. \square

The graph of the function $\nu \mapsto 1 + 2 \frac{(r^2 - \nu^2)J_\nu(r)}{rJ'_\nu(r)} + 2 \left(1 - \frac{1}{\nu}\right) \frac{rJ'_\nu(r)}{J_\nu(r)}$ for $\nu \in \{0.5, 1, 1.5, 2.5\}$ on $[0, 1]$

Theorem 3.2. i. *If $\nu > -1$, then the radius of uniform convexity of the function g_ν is the smallest positive root of the equation*

$$1 + 2r \frac{(2\nu - 1)J_{\nu+1}(r) - rJ_\nu(r)}{J_\nu(r) - rJ_{\nu+1}(r)} = 0.$$

Moreover, $r_\alpha^c(g_\nu) < \alpha_{\nu,1} < j_{\nu,1}$, where $\alpha_{\nu,1}$ is the first positive zero of the Dini function $z \mapsto (1 - \nu)J_\nu(z) + zJ'_\nu(z)$.

ii. *If $\nu \in (-2, -1)$, then the radius of uniform convexity of the function g_ν is $r^{uc}(g_\nu)$, where $r^{uc}(g_\nu)$ is the unique root of the equation*

$$1 + 2r \frac{rI_\nu(r) - (2\nu - 1)I_{\nu+1}(r)}{I_\nu(r) + rI_{\nu+1}(r)} = 0,$$

in the interval $(0, a)$.

Proof. First we prove part *i* for $\nu > -1$ and later part *ii* for $\nu \in (-2, -1)$. In [3, Lemma 2.4] has been proven the equality

$$(3.6) \quad \frac{zg''_\nu(z)}{g'_\nu(z)} = z \frac{zJ_{\nu+2}(z) - 3J_{\nu+1}(z)}{J_\nu(z) - zJ_{\nu+1}(z)} = - \sum_{n \geq 1} \frac{2z^2}{\alpha_{\nu,n}^2 - z^2}$$

where $\alpha_{\nu,n}$ is the n th positive zeros of the Dini function $z \mapsto (1 - \nu)J_\nu(z) + zJ'_\nu(z)$. Using this equality, in [3] the following inequality has been proven

$$(3.7) \quad \operatorname{Re} \left(1 + \frac{zg''_\nu(z)}{g'_\nu(z)} \right) \geq 1 + \frac{rg''_\nu(r)}{g'_\nu(r)}, \quad |z| \leq r < \alpha_{\nu,1}.$$

Equality (3.6) also implies that

$$(3.8) \quad \begin{aligned} \left| \frac{zg''_\nu(z)}{g'_\nu(z)} \right| &= \left| \sum_{n \geq 1} \frac{2z^2}{\alpha_{\nu,n}^2 - z^2} \right| \leq \sum_{n \geq 1} \left| \frac{2z^2}{\alpha_{\nu,n}^2 - z^2} \right| \leq \sum_{n \geq 1} \frac{2r^2}{\alpha_{\nu,n}^2 - r^2} \\ &= - \frac{rg''_\nu(r)}{g'_\nu(r)}, \quad |z| \leq r < \alpha_{\nu,1}. \end{aligned}$$

Now summarizing (3.7) and (3.8) we get

$$(3.9) \quad \operatorname{Re} \left(1 + \frac{zg''_\nu(z)}{g'_\nu(z)} \right) - \left| \frac{zg''_\nu(z)}{g'_\nu(z)} \right| \geq 1 + 2 \frac{rg''_\nu(r)}{g'_\nu(r)}, \quad |z| \leq r < \alpha_{\nu,1}.$$

The equality holds if and only if $z = r$. Finally we get

$$\inf_{|z|<r} \left[\operatorname{Re} \left(1 + \frac{zg''_\nu(z)}{g'_\nu(z)} \right) - \left| \frac{zg''_\nu(z)}{g'_\nu(z)} \right| \right] = 1 + 2 \frac{rg''_\nu(r)}{g'_\nu(r)}, \quad r \in (0, \alpha_{\nu,1}).$$

The mapping $\varphi_\nu : (0, \alpha_{\nu,1}) \rightarrow \mathbb{R}$ defined by $\varphi_\nu(r) = 1 + 2 \frac{rg''_\nu(r)}{g'_\nu(r)} = 1 - 2 \sum_{n \geq 1} \frac{2r^2}{\alpha_{\nu,n}^2 - r^2}$ is strictly decreasing, $\lim_{r \searrow 0} \varphi_\nu(r) = 1$ and $\lim_{r \nearrow \alpha_{\nu,1}} \varphi_\nu(r) = -\infty$. By using the recurrence relation $2\nu J_\nu(z) = z[J_{\nu-1}(z) + J_{\nu+1}(z)]$, it follows that the equation $1 + 2 \frac{rg''_\nu(r)}{g'_\nu(r)} = 1 + 2r \frac{(2\nu-1)J_{\nu+1}(r) - rJ_\nu(r)}{J_\nu(r) - rJ_{\nu+1}(r)} = 0$ has a unique root $r_0 \in (0, \alpha_{\nu,1})$ and $r_0 = r^{uc}(g_\nu)$.

ii. By using the result of Hurwitz [48, p. 305] on zeros of Bessel functions of the first kind, the condition $\nu \in (-2, -1)$ implies $\alpha_{\nu,1} = ia$, $a > 0$ and $\alpha_{\nu,n} > 0$ for $n \in \{2, 3, \dots\}$. Thus, from equality (3.6), we have

$$\begin{aligned} (3.10) \quad 1 + \frac{zg''_\nu(z)}{g'_\nu(z)} &= 1 + \frac{2z^2}{a^2 + z^2} - 2 \sum_{n \geq 2} \frac{z^2}{\alpha_{\nu,n}^2 - z^2} \\ &= 1 - \frac{3a^2}{2(1+\nu)} \frac{z^2}{a^2 + z^2} - 2 \sum_{n \geq 2} \frac{a^2 + \alpha_{\nu,n}^2}{\alpha_{\nu,n}^2} \frac{z^4}{(a^2 + z^2)(\alpha_{\nu,n}^2 - z^2)} \end{aligned}$$

Here, we used following equality (see [5, p. 305])

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_{\nu,n}^2} = \frac{3}{4(\nu+1)} \text{ and so } \frac{1}{a^2} = -\frac{3}{4(\nu+1)} + \sum_{n=2}^{\infty} \frac{1}{\alpha_{\nu,n}^2}.$$

In [5, p. 305] the following equality has been proven

$$\begin{aligned} (3.11) \quad \operatorname{Re} \left(1 + \frac{zg''_\nu(z)}{g'_\nu(z)} \right) &\geq 1 + \frac{3a^2}{2(1+\nu)} \frac{r^2}{a^2 + r^2} \\ &\quad - 2 \sum_{n \geq 2} \frac{a^2 + \alpha_{\nu,n}^2}{\alpha_{\nu,n}^2} \frac{r^4}{(a^2 + r^2)(\alpha_{\nu,n}^2 - r^2)} \\ &= 1 + \frac{irg''_\nu(ir)}{g'_\nu(ir)} > 0, \quad |z| \leq r < a. \end{aligned}$$

On the other hand, if in inequality (2.5) we replace z by z^2 , and b by $\alpha_{\nu,n}$, taking in account that $-\frac{3a^2}{2(1+\nu)} > 0$, we get the following inequalities

$$\begin{aligned} (3.12) \quad \left| \frac{zg''_\nu(z)}{g'_\nu(z)} \right| &= \left| -\frac{3a^2}{2(1+\nu)} \frac{z^2}{a^2 + z^2} - 2 \sum_{n \geq 2} \frac{a^2 + \alpha_{\nu,n}^2}{\alpha_{\nu,n}^2} \frac{z^4}{(a^2 + z^2)(\alpha_{\nu,n}^2 - z^2)} \right| \\ &\leq -\frac{3a^2}{2(1+\nu)} \left| \frac{z^2}{a^2 + z^2} \right| + 2 \sum_{n \geq 2} \frac{a^2 + \alpha_{\nu,n}^2}{\alpha_{\nu,n}^2} \left| \frac{z^4}{(a^2 + z^2)(\alpha_{\nu,n}^2 - z^2)} \right| \\ &\leq -\frac{3a^2}{2(1+\nu)} \frac{r^2}{a^2 - r^2} + 2 \sum_{n \geq 2} \frac{a^2 + \alpha_{\nu,n}^2}{\alpha_{\nu,n}^2} \frac{r^4}{(a^2 - r^2)(\alpha_{\nu,n}^2 + r^2)} \\ &= -\frac{irg''_\nu(ir)}{g'_\nu(ir)}, \quad |z| \leq r < a. \end{aligned}$$

From inequalities (3.11) and (3.12) we get

$$\inf_{|z|<r} \left[\operatorname{Re} \left(1 + \frac{zg''_\nu(z)}{g'_\nu(z)} \right) - \left| \frac{zg''_\nu(z)}{g'_\nu(z)} \right| \right] = 1 + 2 \frac{irg''_\nu(ir)}{g'_\nu(ir)}$$

for every $r \in (0, a)$. Now, consider the function $\Theta_\nu : (0, a) \rightarrow \mathbb{R}$, defined by

$$\Theta_\nu(r) = 1 + 2 \frac{irg_\nu''(ir)}{g_\nu'(ir)} = 1 - \frac{4r^2}{a^2 - r^2} + 4 \sum_{n \geq 2} \frac{r^2}{\alpha_{\nu,n}^2 + r^2}.$$

Since $\Theta_\nu(r) = 1 + 2 \frac{irg_\nu''(ir)}{g_\nu'(ir)} = 1 - \frac{4r^2}{a^2 - r^2} + 4 \sum_{n \geq 2} \frac{r^2}{\alpha_{\nu,n}^2 + r^2}$ is strictly decreasing, $\lim_{r \searrow 0} \Theta_\nu(r) = 1$ and $\lim_{r \nearrow a} \Theta_\nu(r) = -\infty$ it follows that the equation $1 + 2 \frac{irg_\nu''(ir)}{g_\nu'(ir)} = 1 + 2r \frac{rI_\nu(r) - (2\nu-1)I_{\nu+1}(r)}{I_\nu(r) + rI_{\nu+1}(r)} = 0$ has a unique root $r^{uc}(g_\nu) \in (0, a)$. \square

The graph of the function $\nu \mapsto 1 + 2r \frac{(2\nu-1)J_{\nu+1}(r) - rJ_\nu(r)}{J_\nu(r) - rJ_{\nu+1}(r)}$ for $\nu \in \{-0.5, 0, 0.5, 1.5\}$ on $[0, 0.86]$

The graph of
on $[0, 0.5]$

Theorem 3.3. i. *If $\nu > -1$, then the radius of uniform convexity of the function h_ν is the smallest positive root of the equation*

$$1 + r^{\frac{1}{2}} \frac{2(\nu-1)J_{\nu+1}(r^{\frac{1}{2}}) - r^{\frac{1}{2}}J_\nu(r^{\frac{1}{2}})}{2J_\nu(r^{\frac{1}{2}}) - r^{\frac{1}{2}}J_{\nu+1}(r^{\frac{1}{2}})} = 0.$$

Moreover, $r_\alpha^c(h_\nu) < \beta_{\nu,1}^2 < j_{\nu,1}^2$, where $\beta_{\nu,1}$ is the first positive zero of the Dini function $z \mapsto (2-\nu)J_\nu(z) + zJ'_\nu(z)$.

ii. *If $\nu \in (-2, -1)$, then the radius of uniform convexity of the function h_ν is $r^{uc}(g_\nu)$, where $r^{uc}(g_\nu)$ is the unique root of the equation*

$$1 + r^{\frac{1}{2}} \frac{r^{\frac{1}{2}}I_\nu(r^{\frac{1}{2}}) - 2(\nu-1)I_{\nu+1}(r^{\frac{1}{2}})}{2I_\nu(r^{\frac{1}{2}}) + r^{\frac{1}{2}}I_{\nu+1}(r^{\frac{1}{2}})} = 0,$$

in the interval $(0, a)$.

Proof. i. In [3, Lemma 2.5] the following equality has been proven

$$(3.13) \quad \frac{zh_\nu''(z)}{h_\nu'(z)} = \frac{\nu(\nu-2)J_\nu(z^{\frac{1}{2}}) + (3-2\nu)z^{\frac{1}{2}}J'_\nu(z^{\frac{1}{2}}) + zJ''_\nu(z^{\frac{1}{2}})}{2(2-\nu)J_\nu(z^{\frac{1}{2}}) + 2z^{\frac{1}{2}}J'_\nu(z^{\frac{1}{2}})} = - \sum_{n \geq 1} \frac{z}{\beta_{\nu,n}^2 - z},$$

and in the same paper, using the above equality, in Theorem 1.3 the following inequality was deduced

$$(3.14) \quad \operatorname{Re} \left(1 + \frac{zh_\nu''(z)}{h_\nu'(z)} \right) \geq 1 + \frac{rh_\nu''(r)}{h_\nu'(r)} = 1 + \frac{r^{\frac{1}{2}}}{2} \frac{r^{\frac{1}{2}}J_{\nu+2}(r^{\frac{1}{2}}) - 4J_{\nu+1}(r^{\frac{1}{2}})}{2J_\nu(r^{\frac{1}{2}}) - r^{\frac{1}{2}}J_{\nu+1}(r^{\frac{1}{2}})},$$

where $|z| < r < \beta_{\nu,1}^2 < j_{\nu,1}^2$ and $\beta_{\nu,n}$ is the n th positive zeros of the Dini function $z \mapsto (2 - \nu)J_\nu(z) + zJ'_\nu(z)$. The equality (3.13) and the second inequality of (2.4) imply

$$(3.15) \quad \begin{aligned} \left| \frac{zh''_\nu(z)}{h'_\nu(z)} \right| &= \left| \sum_{n \geq 1} \frac{z}{\beta_{\nu,n}^2 - z} \right| \leq \sum_{n \geq 1} \left| \frac{z}{\beta_{\nu,n}^2 - z} \right| \leq \sum_{n \geq 1} \frac{r}{\beta_{\nu,n}^2 - r} \\ &= -\frac{rh''_\nu(r)}{h'_\nu(r)}, \quad |z| < r < \beta_{\nu,1}^2. \end{aligned}$$

From the inequalities (3.14) and (3.15) we infer

$$(3.16) \quad \operatorname{Re} \left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)} \right) - \left| \frac{zh''_\nu(z)}{h'_\nu(z)} \right| \geq 1 + 2r \frac{h''_\nu(r)}{h'_\nu(r)} \quad |z| < r < \beta_{\nu,1}^2.$$

The equality holds if and only if $z = r$. Thus we get

$$\begin{aligned} \inf_{|z| < r} \left[\operatorname{Re} \left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)} \right) - \left| \frac{zh''_\nu(z)}{h'_\nu(z)} \right| \right] &= 1 + 2r \frac{h''_\nu(r)}{h'_\nu(r)} \\ &= 1 + r^{\frac{1}{2}} \frac{r^{\frac{1}{2}} J_{\nu+2}(r^{\frac{1}{2}}) - 4J_{\nu+1}(r^{\frac{1}{2}})}{2J_\nu(r^{\frac{1}{2}}) - r^{\frac{1}{2}} J_{\nu+1}(r^{\frac{1}{2}})}, \end{aligned}$$

for every $r \in (0, \beta_{\nu,1}^2)$. Since the mapping $\phi_\nu : (0, \beta_{\nu,1}^2) \rightarrow \mathbb{R}$ defined by $\phi_\nu(r) = 1 + 2r \frac{h''_\nu(r)}{h'_\nu(r)} = 1 - \sum_{n \geq 1} \frac{2r}{\beta_{\nu,n}^2 - r}$ is strictly decreasing, and $\lim_{r \searrow 0} \phi_\nu(r) = 1$ and $\lim_{r \nearrow \alpha_{\nu,1}} \phi_\nu(r) = -\infty$, it follows that the equation $1 + r^{\frac{1}{2}} \frac{r^{\frac{1}{2}} J_{\nu+2}(r^{\frac{1}{2}}) - 4J_{\nu+1}(r^{\frac{1}{2}})}{2J_\nu(r^{\frac{1}{2}}) - r^{\frac{1}{2}} J_{\nu+1}(r^{\frac{1}{2}})} = 1 + r^{\frac{1}{2}} \frac{2(\nu-1)J_{\nu+1}(r^{\frac{1}{2}}) - r^{\frac{1}{2}} J_\nu(r^{\frac{1}{2}})}{2J_\nu(r^{\frac{1}{2}}) - r^{\frac{1}{2}} J_{\nu+1}(r^{\frac{1}{2}})} = 0$ has a unique root $r_0 \in (0, \beta_{\nu,1}^2)$, and this root is the radius of uniform convexity. In the last equality we use the recurrence relation $2\nu J_\nu(z) = z[J_{\nu-1}(z) + J_{\nu+1}(z)]$.

ii. By using the result of Hurwitz [48, p. 305] on zeros of Bessel functions of the first kind, the condition $\nu \in (-2, -1)$ implies $\beta_{\nu,1} = ib$, $b > 0$ and $0 < \beta_{\nu,2} < \beta_{\nu,3} < \dots \beta_{\nu,n} < \dots$ for $n \in \{2, 3, \dots\}$. Thus, from equality (3.13), we have

$$\begin{aligned} 1 + \frac{zh''_\nu(z)}{h'_\nu(z)} &= 1 + \frac{z}{b^2 + z} - \sum_{n \geq 2} \frac{z}{\beta_{\nu,n}^2 - z} \\ &= 1 - \frac{b^2}{2(1 + \nu)} \frac{z}{b^2 + z} - \sum_{n \geq 2} \frac{b^2 + \beta_{\nu,n}^2}{\beta_{\nu,n}^2} \frac{z^2}{(b^2 + z)(\beta_{\nu,n}^2 - z)}. \end{aligned}$$

Here, we used the following equality (see [5, p. 305])

$$\sum_{n=1}^{\infty} \frac{1}{\beta_{\nu,n}^2} = \frac{1}{2(\nu+1)} \quad \text{and so} \quad \frac{1}{b^2} = \frac{1}{2(\nu+1)} - \sum_{n=2}^{\infty} \frac{1}{\beta_{\nu,n}^2}.$$

In [5, p. 305] the following equality has been proven

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)} \right) &\geq 1 + \frac{b^2}{2(1 + \nu)} \frac{r}{b^2 - r} - \sum_{n \geq 2} \frac{b^2 + \beta_{\nu,n}^2}{\beta_{\nu,n}^2} \frac{r^2}{(b^2 - r)(\beta_{\nu,n}^2 + r)} \\ &= 1 + \frac{-rh''_\nu(-r)}{h'_\nu(-r)} > 0, \quad |z| \leq r < b^2. \end{aligned}$$

On the other hand, from inequality (2.5) we get

$$\begin{aligned}
\left| \frac{zh''_\nu(z)}{h'_\nu(z)} \right| &= \left| -\frac{b^2}{2(1+\nu)} \frac{z}{b^2+z} - \sum_{n \geq 2} \frac{b^2 + \beta_{\nu,n}^2}{\beta_{\nu,n}^2} \frac{z^2}{(b^2+z)(\beta_{\nu,n}^2-z)} \right| \\
&\leq -\frac{b^2}{2(1+\nu)} \left| \frac{z}{b^2+z} \right| + \sum_{n \geq 2} \frac{b^2 + \beta_{\nu,n}^2}{\beta_{\nu,n}^2} \left| \frac{z^2}{(b^2+z)(\beta_{\nu,n}^2-z)} \right| \\
&\leq -\frac{b^2}{2(1+\nu)} \frac{r}{b^2-r} + \sum_{n \geq 2} \frac{b^2 + \beta_{\nu,n}^2}{\beta_{\nu,n}^2} \frac{r^2}{(b^2-r)(\beta_{\nu,n}^2+r)} \\
&= -\frac{rh''_\nu(-r)}{h'_\nu(-r)}, \quad |z| \leq r < b^2.
\end{aligned}$$

Consequently the following inequality holds

$$\inf_{|z| < r} \left[\operatorname{Re} \left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)} \right) - \left| \frac{zh''_\nu(z)}{h'_\nu(z)} \right| \right] = 1 + 2 \frac{-rh''_\nu(-r)}{h'_\nu(-r)}$$

for every $r \in (0, b^2)$. Since the mapping

$$\Phi_\nu : (0, b^2) \rightarrow \mathbb{R}, \quad \Phi_\nu(r) = 1 + 2 \frac{-rh''_\nu(-r)}{h'_\nu(-r)} = 1 - 2 \frac{r}{b^2-r} + 2 \sum_{n \geq 2} \frac{r}{\beta_{\nu,n}^2 + r}$$

is strictly decreasing, $\lim_{r \searrow 0} \Phi_\nu = 1$ and $\lim_{r \nearrow b^2} \Phi_\nu = -\infty$. Thus it follows that the equation $1 + 2 \frac{-rh''_\nu(-r)}{h'_\nu(-r)} = 1 + r^{\frac{1}{2}} \frac{r^{\frac{1}{2}} I_\nu(r^{\frac{1}{2}}) - 2(\nu-1) I_{\nu+1}(r^{\frac{1}{2}})}{2 I_\nu(r^{\frac{1}{2}}) + r^{\frac{1}{2}} I_{\nu+1}(r^{\frac{1}{2}})} = 0$ has a unique root $r^{uc}(h_\nu) \in (0, b^2)$. \square

The graph of the function $\nu \mapsto 1 + r^{\frac{1}{2}} \frac{2(\nu-1) J_{\nu+1}(r^{\frac{1}{2}}) - r^{\frac{1}{2}} J_\nu(r^{\frac{1}{2}})}{2 J_\nu(r^{\frac{1}{2}}) - r^{\frac{1}{2}} J_{\nu+1}(r^{\frac{1}{2}})}$ for $\nu \in \{-0.5, 0, 0.5, 1.5\}$ on $[0, 1]$ The graph of the function $\nu \mapsto 1 + r^{\frac{1}{2}} \frac{2(\nu-1) J_{\nu+1}(r^{\frac{1}{2}}) - r^{\frac{1}{2}} J_\nu(r^{\frac{1}{2}})}{2 J_\nu(r^{\frac{1}{2}}) - r^{\frac{1}{2}} J_{\nu+1}(r^{\frac{1}{2}})}$ for $\nu \in \{-1.8, -1.5, -1.4, -1.3, -1.2, -1.1, -1.0, -0.9, -0.8, -0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5\}$ on $[0, 1]$

3.2. Uniform convexity of normalized Bessel functions. Now, let us recall some results on the geometric behavior of the functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$. In 2010 Szász [43] investigated the starlikeness of $h_\nu(z)$ in case of $\nu \geq \nu_* \simeq -0.5623\dots$, where ν_* is the unique root of the equation $f'_\nu(1) = 0$ in $(-1, 1)$. Baricz and Szász [3] proved that $f_\nu(z)$ and $g_\nu(z)$ are convex in $U \Leftrightarrow \nu \geq 1$, and $h_\nu(z)$ is convex in $U \Leftrightarrow \nu \geq \nu_{**} \simeq -0.1438\dots$, where ν_{**} is the unique root of the equation $(2\nu-4)J_{\nu+1}(1) + 3J_\nu(1) = 0$. Furthermore, Baricz and Szász [6], Baricz et al. [7] and Baricz et al. [8] obtained necessary and sufficient conditions for the starlikeness and close-to-convexity of the function $h_\nu(z)$ and its derivatives, some special combinations of Bessel functions and their derivatives, and the functions $f_\nu(z)$,

$g_\nu(z)$ and derivatives of $h_\nu(z)$ in U , respectively, by using a result of Shah and Trimble (see [38, Theorem 2]) about transcendental entire functions with univalent derivatives. In this section, we deal with the uniform convexity of the normalized Bessel functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$ in U .

Theorem 3.4. *The function f_ν is uniformly convex in U if and only if $\nu > \nu_1 \simeq 1.4426\dots$, where ν_1 is the unique root of the equation*

$$\nu(3\nu - 2)(J_\nu(1))^2 + \nu(4\nu - 5)J_\nu(1)J_{\nu-1}(1) + 2(1 - \nu)(J_{\nu-1}(1))^2 = 0$$

situated in (ν^, ∞) , where $\nu^* \simeq 0.39001\dots$ is the root of the equation $J'_\nu(1) = 0$.*

Proof. According to (3.5) for $z \in U$ we know that

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)} \right) - \left| \frac{zf''_\nu(z)}{f'_\nu(z)} \right| &\geq 1 - 2 \sum_{n \geq 1} \left(\frac{2r^2}{j_{\nu,n}^2 - r^2} - \left(1 - \frac{1}{\nu} \right) \frac{2r^2}{j_{\nu,n}^2 - r^2} \right) \\ &= 1 + 2 \frac{rJ''_\nu(r)}{J'_\nu(r)} + 2 \left(\frac{1}{\nu} - 1 \right) \frac{rJ'_\nu(r)}{J_\nu(r)} = 1 + 2 \frac{rf''_\nu(r)}{f'_\nu(r)}. \end{aligned}$$

Since the mapping $\psi_\nu : (0, j'_{\nu,1}) \rightarrow \mathbb{R}$ defined by $\psi_\nu(r) = 1 + 2 \frac{rf''_\nu(r)}{f'_\nu(r)}$ is strictly decreasing and the inequalities $1 < j'_{\nu,n} < j_{\nu,n}$ hold for $n \in \{1, 2, \dots\}$, it follows that

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)} \right) - \left| \frac{zf''_\nu(z)}{f'_\nu(z)} \right| &\geq 1 - 2 \sum_{n \geq 1} \left(\frac{2r^2}{j_{\nu,n}^2 - r^2} - \left(1 - \frac{1}{\nu} \right) \frac{2r^2}{j_{\nu,n}^2 - r^2} \right) \\ &\geq 1 - 2 \sum_{n \geq 1} \left(\frac{2}{j_{\nu,n}^2 - 1} - \left(1 - \frac{1}{\nu} \right) \frac{2}{j_{\nu,n}^2 - 1} \right) \\ &= 1 + 2 \frac{J''_\nu(1)}{J'_\nu(1)} + 2 \left(\frac{1}{\nu} - 1 \right) \frac{J'_\nu(1)}{J_\nu(1)} = 1 + 2 \frac{f''_\nu(1)}{f'_\nu(1)}. \end{aligned}$$

Now, consider the function $\psi : (\nu^*, \infty) \rightarrow \mathbb{R}$, defined by

$$\psi(\nu) = 1 + 2 \frac{f''_\nu(1)}{f'_\nu(1)} = 1 + 2 \frac{J''_\nu(1)}{J'_\nu(1)} + 2 \left(\frac{1}{\nu} - 1 \right) \frac{J'_\nu(1)}{J_\nu(1)}.$$

We know that $\nu \mapsto 1 + \frac{f''_\nu(1)}{f'_\nu(1)} = 1 + \frac{J''_\nu(1)}{J'_\nu(1)} + \left(\frac{1}{\nu} - 1 \right) \frac{J'_\nu(1)}{J_\nu(1)}$ is strictly increasing on (ν^*, ∞) (see [3]), thus the function ψ is strictly increasing on (ν^*, ∞) too. Consequently, if $\nu > \nu_1$, then we get the inequality $\psi(\nu) \geq \psi(\nu_1)$. This in turn implies that ν_1 is the smallest value having the property that the condition $\nu \geq \nu_1$ implies that for all $z \in U$ we have

$$\operatorname{Re} \left(1 + \frac{zf''_\nu(z)}{f'_\nu(z)} \right) - \left| \frac{zf''_\nu(z)}{f'_\nu(z)} \right| > \psi(\nu_1) = 0.$$

Thus, we proved that the function f_ν is uniformly convex in U if and only if $\nu > \nu_1$, where ν_1 is the unique root of the equation

$$1 + 2 \frac{J''_\nu(1)}{J'_\nu(1)} + 2 \left(\frac{1}{\nu} - 1 \right) \frac{J'_\nu(1)}{J_\nu(1)} = 0.$$

The function $J_\nu(z)$ satisfies the Bessel differential equation $z^2 w''(z) + zw'(z) + (1 - \nu^2)w(z) = 0$ and by using the recurrence relation $zJ'_\nu(z) = zJ_{\nu-1}(z) - \nu J_\nu(z)$ the above equation can be rewritten as follows

$$\nu(3\nu - 2)(J_\nu(1))^2 + \nu(4\nu - 5)J_\nu(1)J_{\nu-1}(1) + 2(1 - \nu)(J_{\nu-1}(1))^2 = 0.$$

In last equality, we used $J_\nu(1) > 0$ and $J'_\nu(1) > 0$ when $\nu > \nu^*$ (see [3]). \square

Theorem 3.5. *The function g_ν is uniformly convex in U if and only if $\nu > \nu_2 \simeq 2.44314...$, where ν_2 is the unique root of the equation*

$$(4\nu - 3)J_{\nu+1}(1) - J_\nu(1) = 0$$

situated in $[0, \infty)$.

Proof. Taking into account the the inequality (3.9) for $z \in U$ we know that

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{zg_\nu''(z)}{g_\nu'(z)} \right) - \left| \frac{zg_\nu''(z)}{g_\nu'(z)} \right| &\geq 1 - 4 \sum_{n \geq 1} \frac{r^2}{\alpha_{\nu,n}^2 - r^2} \\ &= 1 + 2 \frac{rg_\nu''(r)}{g_\nu'(r)}, \quad |z| \leq r < \alpha_{\nu,1}. \end{aligned}$$

Since the mapping $\varphi_\nu : [0, \alpha_{\nu,1}) \rightarrow \mathbb{R}$ defined by $\varphi_\nu(r) = 1 + 2 \frac{rg_\nu''(r)}{g_\nu'(r)}$ is strictly decreasing and the inequalities $1 < \alpha_{\nu,1}$ hold for $\nu \geq 0$ (see [3, Lemma 2.4]), we get that

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{zg_\nu''(z)}{g_\nu'(z)} \right) - \left| \frac{zg_\nu''(z)}{g_\nu'(z)} \right| &\geq 1 - 4 \sum_{n \geq 1} \frac{r^2}{\alpha_{\nu,n}^2 - r^2} \\ &\geq 1 - 4 \sum_{n \geq 1} \frac{1}{\alpha_{\nu,n}^2 - 1} = 1 + 2 \frac{g_\nu''(1)}{g_\nu'(1)} \\ &= 1 + 2 \frac{(2\nu - 1)J_{\nu+1}(1) - J_\nu(1)}{J_\nu(1) - J_{\nu+1}(1)}. \end{aligned}$$

Now, consider the function $\varphi : [0, \infty) \rightarrow \mathbb{R}$, defined by

$$\varphi(\nu) = 1 + 2 \frac{(2\nu - 1)J_{\nu+1}(1) - J_\nu(1)}{J_\nu(1) - J_{\nu+1}(1)}.$$

We know that $\nu \mapsto 1 + \frac{g_\nu''(1)}{g_\nu'(1)} = 1 + \frac{(2\nu - 1)J_{\nu+1}(1) - J_\nu(1)}{J_\nu(1) - J_{\nu+1}(1)}$ is stricly increasing and $J_\nu(1) - J_{\nu+1}(1) > 0$ on $[0, \infty)$ (see [3]), thus the function φ is stricly increasing on $[0, \infty)$ too. Since φ is stricly increasing, it follows that if $\nu > \nu_2 \simeq 2.44314...$, then we get the inequality

$$1 + 2 \frac{(2\nu - 1)J_{\nu+1}(1) - J_\nu(1)}{J_\nu(1) - J_{\nu+1}(1)} = \varphi(\nu) \geq \varphi(\nu_2) = 1 + 2 \frac{(2\nu_2 - 1)J_{\nu_2+1}(1) - J_{\nu_2}(1)}{J_{\nu_2}(1) - J_{\nu_2+1}(1)} = 0.$$

Thus, under the conditon $\nu > \nu_2 \simeq 2.44314...$, we have for $z \in U$

$$\operatorname{Re} \left(1 + \frac{zg_\nu''(z)}{g_\nu'(z)} \right) - \left| \frac{zg_\nu''(z)}{g_\nu'(z)} \right| > \varphi(\nu_2) = 0.$$

Consequently, we proved that the function g_ν is uniformly convex in U if and only if $\nu > \nu_2 \simeq 2.44314...$, where ν_2 is the unique root of the equation

$$1 + 2 \frac{(2\nu - 1)J_{\nu+1}(1) - J_\nu(1)}{J_\nu(1) - J_{\nu+1}(1)} = 0.$$

In [3], the authors proved that $J_\nu(1) - J_{\nu+1}(1) > 0$ when $\nu \geq 0$. Thus the proof is completed. \square

Theorem 3.6. *The function h_ν is uniformly convex in U if and only if $\nu > \nu_3 \simeq 0.30608...$, where ν_3 is the unique root of the equation*

$$(2\nu - 3)J_{\nu+1}(1) + J_\nu(1) = 0$$

situated in $[0, \infty)$.

Proof. Taking into account the the inequality (3.16) for $z \in U$ we know that

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)} \right) - \left| \frac{zh''_\nu(z)}{h'_\nu(z)} \right| &\geq 1 - \sum_{n \geq 1} \frac{2r}{\beta_{\nu,n}^2 - r} \\ &= 1 + 2 \frac{rh''_\nu(r)}{h'_\nu(r)}, \quad |z| < r < \beta_{\nu,1}^2 < j_{\nu,1}^2. \end{aligned}$$

Since the mapping $\phi_\nu : [0, \alpha_{\nu,1}) \rightarrow \mathbb{R}$ defined by $\phi_\nu(r) = 1 + 2 \frac{rh''_\nu(r)}{h'_\nu(r)}$ is strictly decreasing and the inequalities $1 < \beta_{\nu,1}$ hold for $\nu \geq 0$ (see [3, Lemma 2.4]), we get that

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)} \right) - \left| \frac{zh''_\nu(z)}{h'_\nu(z)} \right| &\geq 1 - \sum_{n \geq 1} \frac{2r}{\beta_{\nu,n}^2 - r} \\ &\geq 1 - \sum_{n \geq 1} \frac{2}{\beta_{\nu,n}^2 - 1} = 1 + 2 \frac{h''_\nu(1)}{h'_\nu(1)} \\ &= 1 + \frac{2(\nu-1)J_{\nu+1}(1) - J_\nu(1)}{2J_\nu(1) - J_{\nu+1}(1)}. \end{aligned}$$

Now, consider the function $\phi : [0, \infty) \rightarrow \mathbb{R}$, defined by

$$\phi(\nu) = 1 + \frac{2(\nu-1)J_{\nu+1}(1) - J_\nu(1)}{2J_\nu(1) - J_{\nu+1}(1)}.$$

We know that $\nu \mapsto 1 + \frac{h''_\nu(1)}{h'_\nu(1)} = 1 + \frac{2(\nu-1)J_{\nu+1}(1) - J_\nu(1)}{2J_\nu(1) - J_{\nu+1}(1)}$ is stricly increasing and $2J_\nu(1) - J_{\nu+1}(1) > 0$ on $[0, \infty)$ (see [3]), thus we can easily see that the function ϕ is stricly increasing on $[0, \infty)$ too. Since ϕ is stricly increasing, it follows that if $\nu > \nu_3 \simeq 0.30608\dots$, then we get the inequality

$$1 + \frac{2(\nu-1)J_{\nu+1}(1) - J_\nu(1)}{2J_\nu(1) - J_{\nu+1}(1)} = \phi(\nu) \geq \phi(\nu_3) = 1 + \frac{2(\nu_3-1)J_{\nu_3+1}(1) - J_{\nu_3}(1)}{2J_{\nu_3}(1) - J_{\nu_3+1}(1)} = 0.$$

Thus, under the conditon $\nu > \nu_3 \simeq 0.30608\dots$, we have for $z \in U$

$$\operatorname{Re} \left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)} \right) - \left| \frac{zh''_\nu(z)}{h'_\nu(z)} \right| > \phi(\nu_3) = 0.$$

Thus, we proved that the function h_ν is uniformly convex in U if and only if $\nu > \nu_3 \simeq 0.30608\dots$, where ν_3 is the unique root of the equation

$$1 + \frac{2(\nu-1)J_{\nu+1}(1) - J_\nu(1)}{2J_\nu(1) - J_{\nu+1}(1)} = 0.$$

In [3], authors proved that $2J_\nu(1) - J_{\nu+1}(1) > 0$ when $\nu \geq 0$. Thus from last equality we obtain

$$(2\nu-3)J_{\nu+1}(1) + J_\nu(1) = 0.$$

□

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